Lösung WS 11/12:

A1)

a) $y''' + 2y'' + 50y' = 30x^2 - 42\sin 7x + 56e^{-x}\cos 7x$ $\quad P(\alpha) = \alpha^3 + 2\alpha^2 + 50\alpha = 0$

$\Leftrightarrow \alpha_1 = 0 \;;\; \alpha_{2/3} = -1 \pm 7i \;;\; y_H = C_1 + e^{-x}\cdot\left(C_2\cos 7x + C_3\sin 7x\right)$

b) $g_1(x) = 30x^2$**:** Resonanz zu $\alpha_1 = 0$ (<y-Glied> fehlt in Dgl!) $\Rightarrow y_{P_1} = x\cdot\left(Ax^2 + Bx + C\right)$

$y_{P_1}' = 3Ax^2 + 2Bx + C \;;\; y_{P_1}'' = 6Ax + 2B \;;\; y_{P_1}''' = 6A$

in Dgl: $6A + 12Ax + 4B + 150Ax^2 + 100Bx + 50C \doteq 30x^2 \Rightarrow A = \frac{1}{5};\; B = -\frac{3}{125};\; C = -\frac{69}{3125}$

$y_{P_1} = \frac{x^3}{5} - \frac{3x^2}{125} - \frac{69x}{3125}$

$g_2(x) = -42\cdot\sin 7x$**:** keine Resonanz! $\Rightarrow y_{P_2} = \mathrm{Im}\left[\frac{-42e^{7ix}}{P(7i)}\right] = -42\cdot\mathrm{Im}\left[\frac{\cos 7x + i\cdot\sin 7x}{7(-14+i)}\right]$

$= -6\cdot\mathrm{Im}\left[\frac{(\cos 7x + i\cdot\sin 7x)\cdot(-14-i)}{(-14+i)\cdot(-14-i)}\right] = \frac{6}{197}\left(\cos 7x + 14\cdot\sin 7x\right)$

$g_3(x) = 56e^{-x}\cdot\cos 7x$**:** Resonanz zu $\alpha_2 = -1+7i \Rightarrow y_{P_3} = \mathrm{Re}\left[\frac{56e^{-(1+7i)x}\cdot x}{P'(-1+7i)}\right]$

$P'(\alpha) = 3\alpha^2 + 4\alpha + 50 \;;\; P'(-1+7i) = 3(-1+7i)^2 + 4(-1+7i) + 50 = -14\cdot(7+i)$

$y_{P_3} = \frac{56x}{-14}\cdot e^{-x}\cdot\mathrm{Re}\left[\frac{(\cos 7x + i\cdot\sin 7x)\cdot(7-i)}{(7+i)\cdot(7-i)}\right] = -\frac{2}{25}x\cdot e^{-x}\cdot\left(7\cos 7x + \sin 7x\right)$

A2:

a) $y''\cdot\ln x - \frac{1}{x}\cdot y' = x\cdot\ln^2 x \;,\; x > 0$ $\Rightarrow$ Homogene Lösung: Achtung: $y'' = f(x, y')$

Ansatz $y' = p$**:** $\Rightarrow \frac{dp}{dx}\cdot\ln x - \frac{1}{x}\cdot p = 0 \Leftrightarrow \frac{dp}{p} = \frac{\frac{1}{x}dx}{\ln x} \Leftrightarrow \ln|p| = \ln|\ln x| + \ln|C|$

$p = C_1\cdot\ln x$, *also* $y_H = C_1\cdot x(\ln x - 1) + C_2$

b) Inhomogene Lösung (VdK): Normierung der Dgl beachten!!

$$\Rightarrow y'' - \frac{1}{x\cdot\ln x}y' = x\cdot\ln x \quad ; \quad \textit{Wronski-Det.:} \quad W(x) = \begin{vmatrix} x(\ln x - 1) & 1 \\ \ln x & 0 \end{vmatrix} = \boxed{-\ln x} \quad (x \neq 1)$$

$$C_1'(x) = -\frac{1}{\ln x}\cdot\begin{vmatrix} 0 & 1 \\ x\cdot\ln x & 0 \end{vmatrix} = x \quad \Rightarrow \boxed{C_1(x) = \frac{x^2}{2} + K_1}$$

$$C_2'(x) = -\frac{1}{\ln x}\cdot\begin{vmatrix} x(\ln x - 1) & 0 \\ \ln x & x\cdot\ln x \end{vmatrix} = -x^2(\ln x - 1) \quad \Rightarrow \boxed{C_2(x) = \frac{4x^3}{9} - \frac{x^3\ln x}{3} + K_2}$$

$$\textit{also:} \quad y_{allg} = \left(\frac{x^2}{2} + K_1\right)\cdot x(\ln x - 1) + \frac{x^3}{9}(4 - 3\ln x) + K_2 = \boxed{\frac{x^3}{18}(3\ln x - 1) + K_1 x(\ln x - 1) + K_2.}$$

c) **Randwertproblem:**

$$\text{wegen} \quad \lim_{x\to 0}(x^n\cdot\ln x) = 0: \quad \boxed{K_2 = 0} \quad ; \quad y(1) = 0 = -\frac{1}{18} - K_1 \quad \Leftrightarrow \boxed{K_1 = -\frac{1}{18}}$$

$$\Rightarrow \boxed{y_{spez} = \frac{x}{18}\left(x^2\cdot(3\ln x - 1) - \ln x + 1\right)}$$

d) Taylor-Entwicklung um x = 1:

1) $\boxed{y(1) = 0}$

2) $y' = \frac{1}{18}\left(3x^2\cdot\ln x - x^2 - \ln x + 1\right) + \frac{x}{18}\left(6x\cdot\ln x + 3x - 2x - \frac{1}{x}\right) = \frac{\ln x}{18}(9x^2 - 1) \Rightarrow \boxed{y'(1) = 0}$

3) $y'' = \frac{1}{18}\left(9x - \frac{1}{x} + \ln x\cdot 18x\right) \quad \Rightarrow \boxed{y''(1) = \frac{4}{9}}$

4) $y''' = \frac{1}{18}\left(9 + \frac{1}{x^2} + 18\ln x + 18\right) = \left(\ln x + \frac{3}{2} + \frac{1}{18x^2}\right) \quad \Rightarrow \boxed{y'''(1) = \frac{14}{9}}$

5) $y^{(4)} = \frac{1}{x} - \frac{1}{9x^3} \quad \Rightarrow \boxed{y^{(4)}(1) = \frac{8}{9}} \quad ; \quad y^{(5)} = -\frac{1}{x^2} + \frac{1}{3x^4} \quad \Rightarrow \boxed{y^{(5)}(1) = -\frac{2}{3}}$

$$\textit{also:} \quad f(x) = 0 + 0 + \frac{\frac{4}{9}}{2!}(x-1)^2 + \frac{\frac{14}{9}}{3!}(x-1)^3 + \frac{\frac{8}{9}}{4!}(x-1)^4 + \frac{\frac{-2}{3}}{5!}(x-1)^5 \pm \ldots =$$

$$= \boxed{\frac{2}{9}(x-1)^2 + \frac{7}{27}(x-1)^3 + \frac{1}{27}(x-1)^4 - \frac{1}{180}(x-1)^5 \pm \ldots}$$

A3:

a) $\boxed{A = \begin{pmatrix} 1 & -3 & 1 \\ 2 & -4 & 1 \\ 2 & -3 & 0 \end{pmatrix}} \quad \Rightarrow \det(A - \lambda E) = \begin{vmatrix} 1-\lambda & -3 & 1 \\ 2 & 4-\lambda & 1 \\ 2 & -3 & -\lambda \end{vmatrix} = -(1+\lambda)^3 = 0 \quad \Rightarrow \boxed{\lambda_{1/2/3} = -1}$

$\boxed{\lambda_1 = -1:} \quad \begin{pmatrix} 2 & -3 & 1 \\ 2 & -3 & 1 \\ 2 & -3 & 1 \end{pmatrix} \sim \begin{pmatrix} 2 & -3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad rg = 1, \quad \textit{Rangabfall} = 2 \quad \Rightarrow$ 2 lu EV wählen:

$$\boxed{\vec{a}_{11}=\begin{pmatrix}1\\1\\1\end{pmatrix};\quad \vec{a}_{12}=\begin{pmatrix}0\\1\\3\end{pmatrix}}$$

Damit fehlt nur noch 1 EV: $\vec{y}_3=\left(\vec{a}_1\cdot x+\vec{b}\right)\cdot e^{-x}$

Linearkombination der beiden EV: $\vec{a}_1=u\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}+v\cdot\begin{pmatrix}0\\1\\3\end{pmatrix}=\begin{pmatrix}u\\u+v\\u+3v\end{pmatrix}$

$$\left(\begin{array}{ccc|c}2&-3&1&u\\2&-3&1&u+v\\2&-3&1&u+3v\end{array}\right)\sim\left(\begin{array}{ccc|c}2&-3&1&u\\0&0&0&v\\0&0&0&3V\end{array}\right)\Rightarrow v=0\quad\Rightarrow\vec{a}_1=\begin{pmatrix}1\\1\\1\end{pmatrix}.$$

$$\Rightarrow\vec{b}=\begin{pmatrix}0\\0\\1\end{pmatrix}\quad\Leftrightarrow\boxed{\vec{y}_H=e^{-x}\cdot\left(C_1\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}+C_2\cdot\begin{pmatrix}0\\1\\3\end{pmatrix}+C_3\cdot\begin{pmatrix}x\\x\\x+1\end{pmatrix}\right)}$$

Kontrolle für $y_1'=y_1-3y_2+y_3$:

$$y_1'=e^{-x}\left\{(C_1+C_3x)-3C_1-3C_2-3C_3x+C_1+3C_2+C_3(x+1)\right\}=e^{-x}\left\{-C_1+C_3(-x+1)\right\}$$

$$y_1'=\left((C_1+C_3x)e^{-x}\right)'=-C_1\cdot e^{-x}+C_3\cdot(-x+1)e^{-x}\qquad\square$$

c) Inhomogene Lösung:

$$\boxed{f(x)=\begin{pmatrix}2\\-4\\6\end{pmatrix}\cdot e^{-2x}}\;;\quad\text{keine Resonanz}\quad\Rightarrow\vec{y}_P=\begin{pmatrix}a\\b\\c\end{pmatrix}e^{-2x}\;;\quad\vec{y}_P{}'=\begin{pmatrix}-2a\\-2b\\-2c\end{pmatrix}e^{-2x}$$

$$\vec{y}'=A\cdot\vec{y}+\begin{pmatrix}2\\-4\\6\end{pmatrix}\cdot e^{-2x}\quad\Rightarrow\begin{pmatrix}-2a\\-2b\\-2c\end{pmatrix}e^{-2x}=\underbrace{\begin{pmatrix}1&-3&1\\2&-4&1\\2&-3&0\end{pmatrix}\cdot\begin{pmatrix}a\\b\\c\end{pmatrix}}_{\begin{pmatrix}a-3b+c\\2a-4b+c\\2a-3b\end{pmatrix}}e^{-2x}+\begin{pmatrix}2\\-4\\6\end{pmatrix}\cdot e^{-2x}$$

$$\left(\begin{array}{ccc|c}3&-3&1&-2\\2&-2&1&4\\2&-3&2&-6\end{array}\right)\sim\left(\begin{array}{ccc|c}1&0&-1&4\\0&1&-1&10\\0&-3&4&-14\end{array}\right)\sim\left(\begin{array}{ccc|c}1&0&0&20\\0&1&0&26\\0&0&1&16\end{array}\right)\Rightarrow\boxed{\vec{y}_P=\begin{pmatrix}20\\26\\16\end{pmatrix}\cdot e^{-2x}}$$

$$\boxed{\vec{y}_{allg}=e^{-x}\cdot\left\{C_1\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}+C_2\cdot\begin{pmatrix}0\\1\\3\end{pmatrix}+C_3\cdot\begin{pmatrix}x\\x\\x+1\end{pmatrix}\right\}+\begin{pmatrix}20\\26\\16\end{pmatrix}\cdot e^{-2x}}$$